§4 CONNECTIONS IN LINE BUNDLES

The concept of a connection on a line bundle and – more generally – on a vector bundle is fundamental for geometric considerations in such bundles. A connection is the basic geometric structure on a bundle. Moreover, connections on line bundles are needed in the program of geometric quantization.

A connection on a vector bundle induces a connection form on the corresponding frame bundle; – $L^\times$ in the case of a line bundle L. Conversely, a connection form on a principal fibre bundle P over M with structure group G induces a connection on all associated vector bundles $E_\varrho := P \times_\varrho \mathbb{C}^r$, where $\varrho : G \to GL(r, \mathbb{C})$ is a representation.

In the "ansatz" of geometric quantization presented in §2 together with our understanding of sections in a line bundle we are led to replace $\mathcal{E}(M)$ with the space $\Gamma(M, L)$ of sections in a line bundle L. But this leads (according to §2) immediately to the question of what the directional derivatives should be in the generalised context of line bundles. Geometry gives the answer: The Lie derivatives L_X on $\mathcal{E}(M)$ are replaced by covariant derivatives ∇_X for vector fields $X \in \mathcal{V}(M)$.

(4.1) DEFINITION: A CONNECTION in (or on) a line bundle $\pi : L \to M$ over a manifold M is a collection of maps

$$\nabla : \mathcal{V}(U) \longrightarrow \mathrm{End}_{\mathbb{C}}(\Gamma(U,L)) \ , \ X \longmapsto \nabla_X,$$

$U \subset M$ open, which are compatible with restrictions to open $V \subset U$, such that the following properties are satisfied:

(K1) ∇ is an $\mathcal{E}(U)$ module homomorphism, i.e.

$$\nabla_{fX+Y} = f\nabla_X + \nabla_Y \quad \text{for all } X, Y \in \mathcal{V}(U) \text{ and } f \in \mathcal{E}(U)$$

(K2) $\nabla_X(fs) = (L_X f)\, s + f\, \nabla_X s$, for $X \in \mathcal{V}(U), s \in \Gamma(U,L), f \in \mathcal{E}(U)$.

A connection in the above form is also called a KOSZUL CONNECTION.

Observation: We see immediately, that on the trivial bundle

$$L = M \times \mathbb{C},$$

where we have $\Gamma(U,L) \cong \mathcal{E}(U)$ for open $U \subset M$ (see the discussion after Def. (3.3)) the Lie derivative $L_X : \mathcal{E}(U) \to \mathcal{E}(U)$ is an example of a connection, since $L_{X+Y} = L_X + L_Y$ & $L_{fX} = fL_X$ (K1), and (K2): $L(fg) = L_X f\, g + f\, L_X g$. Are there more connections on $M \times \mathbb{C}$? How do they look?

Every section $s \in \Gamma(M, M\times\mathbb{C})$ has the form $s(a) = (a, f(a))$, $a \in M$, with a uniquely defined function $f \in \mathcal{E}(M)$. With respect to the previously defined section $s_1 \in \Gamma(M,L)$

$$s_1(a) = (a,1),\ a \in M,$$

we have $s = fs_1$. A given connection ∇, therefore, defines, in particular, a map

$$\beta : \mathcal{V}(M) \longrightarrow \mathcal{E}(M)$$

by the unique function $\beta(X) \in \mathcal{E}(M)$ with $\nabla_X s_1 = \beta(X) s_1$. By (K1) β is $\mathcal{E}(M)$-linear:

$$\beta(fX+Y)s_1 = \nabla_{fX+Y} s_1 = f\nabla_X s_1 + \nabla_Y s_2 = f\beta(X) + \beta(Y).$$

Therefore, β is a 1-form $\beta \in \Omega^1(M)$.

(4.2) <u>LEMMA</u>: Each connection ∇ on $M\times\mathbb{C} = L \to M$ is of the form

$$\nabla_X s = \left(L_X f + 2\pi i\, i_X \alpha f\right) s_1, \text{ if } s = fs_1|_U,\ f \in \mathcal{E}(U),$$

with a suitable $\alpha \in \Omega^1(M)$. Conversely, for each $\alpha \in \Omega^1(M)$ the above formula defines a connection on L.

□ <u>Proof</u>. $\nabla_X s = \nabla_X(fs_1) = (L_X f)s_1 + f\beta(X)s_1$ according to (K2), and with $\alpha := \frac{1}{2\pi i}\beta$ we get the corresponding form. Conversely, for any $\alpha \in \Omega^1(M)$

$$\nabla_X s := \left(L_X f + 2\pi i\, i_X\alpha f\right) s_1,\ s = fs_1 \in \Gamma(U,L)\ \&\ X \in \mathcal{V}(U),$$

defines a connection on L:

(K1) is evident. For (K2): $g \in \mathcal{E}(U)$ and $s = f s_1$, $f \in \mathcal{E}(U)$.

$$\nabla_X (gs) = \nabla_X (gf) s_1 = \left(L_X(gf) + 2\pi i \, i_X\alpha \, (gf) \right) s_1$$

$$= \left((L_X g) f + g L_X f + g \, i_X\alpha f \right) s_1 = (L_X g) s + g \nabla_X s . \qquad \square$$

Note, that

$$\Gamma(U, T^*M \otimes L) \cong \mathcal{V}^*(U) \otimes \Gamma(U,L) \cong \mathrm{Hom}_{\mathcal{E}(U)} \left(\mathcal{V}(U), \Gamma(U,L) \right).$$

And a connection according to definition (4.1) can also be described by a $\mathbb{C}$-linear map

$$\nabla : \Gamma(U,L) \longrightarrow \Gamma(U, T^*M \otimes L)$$

with

$$\nabla(fs) = df \otimes s + f \nabla s \quad , \; s \in \Gamma(U,L), \; f \in \mathcal{E}(U),$$

compatible with restrictions.

We now want to describe any connection on a non-trivial line bundle $L \to M$ locally by using (4.2). Let (U_j) be an open cover of M with local trivializations $\varphi_j : L_{U_j} \xrightarrow{\sim} U_j \times \mathbb{C}$ and corresponding transition functions $g_{jk} \in \mathcal{E}(U_j, \mathbb{C}^\times)$. For each $U_j \times \mathbb{C}$ there exists $\alpha_j \in \Omega^1(U_j)$ such that

$$\nabla_X |_{U_j} (f s_j) = \left(L_X f + 2\pi i \, \alpha_j(X) f \right) s_j \qquad (4.2)$$

and we want to see the interrelations of α_j and α_k

on a nonempty intersection $U_{jk} = U_j \cap U_k \neq \emptyset$:

For any section $s \in \Gamma(U, L)$ we have

$$s|_{U \cap U_{jk}} = f_j s_j = f_k s_k \quad \text{with } f_j, f_k \in \mathcal{E}(U_{jk} \cap U).$$

From

$$\nabla_X s = \left(L_X f_j + 2\pi i \alpha_j(X) f_j\right) s_j = \left(L_X f_k + 2\pi i \alpha_k(X) f_k\right) s_k$$

and $s_j = g_{kj} s_k$, $f_j = g_{jk} f_k$ we get

$$\left((L_X g_{jk}) f + g_{jk} L_X f_k + 2\pi i \alpha_j(X) g_{jk} f_k\right) g_{kj} = L_X f_k + 2\pi i \alpha_k(X) f_k$$

Hence, $\left(dg_{jk} g_{kj} + 2\pi i \alpha_j(X)\right) f_k = 2\pi i \alpha_k(X) f_k$, which implies

$$[Z] \quad \alpha_k - \alpha_j = \frac{1}{2\pi i} dg_{jk} g_{jk}^{-1} = \frac{dg_{jk}}{2\pi i g_{jk}} \text{ on } U_{jk} \neq \emptyset.$$

(4.3) <u>PROPOSITION</u>: Let (U_j) be an open cover such that the line bundle $L \to M$ has trivializations over U_j with transition functions $g_{jk} \in \mathcal{E}(U_{jk}, \mathbb{C}^\times)$. Any connection ∇ on L determines uniquely a collection (α_j) of 1-forms $\alpha_j \in \mathcal{E}(U_j)$ with [Z]. Conversely, (α_j) with [Z] induces a connection on L.

□ Indeed, given $s \in \Gamma(U, L)$ with $s|_{U_j} = f_j s_j$, $f_j \in \mathcal{E}(U \cap U_j)$, by

$$\nabla_X f_j s_j := \left(L_X f_j + 2\pi i \alpha_j(X) f_j\right) s_j$$

a connection is defined. □

In a next step we want to understand how every line bundle connection is induced by a global object on the corresponding frame bundle $L^\times \subset L$.

$L^\times$ is simply the bundle $L^\times := \{ l \in L : l \neq 0_a \,, a = \pi(l) \}$. $= L \setminus \{ 0_a \in L_a : a \in M \}$. $L^\times$ is a principal fibre bundle with structure group $\mathbb{C}^\times$. We have the projection

$$\pi : L^\times \longrightarrow M,$$

which is the restriction of the projection $L \to M$, and we have the right action of $\mathbb{C}^\times$

$$\Psi : L^\times \times \mathbb{C}^\times \longrightarrow L^\times \,, \ (l,g) \longrightarrow lg = \Psi(l,g),$$

where $lg = gl$ is simply the multiplication in the fibre L_a, $a = \pi(l)$. $L^\times$ has the trivializations

$$\varphi_j : L^\times_{U_j} \to U_j \times \mathbb{C}^\times,$$

the restrictions of the trivializations $L_{U_j} \to U_j \times \mathbb{C}$ of the original line bundle $L \to M$.

Now, let us consider a connection ∇ on L given by the one forms $\alpha_j \in \Omega^1(U_j)$ with [Z]. Then the forms α_j can be lifted to $L^\times_{U_j}$ by π to yield

$$\pi^*(\alpha_j) \in \Omega^1(L^\times_{U_j}) \,, \ j \in I$$

[Recall: Given a smooth map $\psi : N \to M$ between manifolds N and M the pull back $\varphi^*: \Omega^1(M) \to \Omega^1(N)$ is defined as follows: For $\beta \in \Omega^1(M)$ and $Y \in \mathcal{V}(N)$ we set

$$\varphi^*\beta(Y) := \beta(T\varphi(Y)),$$

where $T\varphi : TN \to TM$ is the derivative of φ. For a tangent vector $\xi = [\gamma(t)]_b \in T_bN$, where $\gamma : I \to N$ is a curve in N through b, $\gamma(0) = b$, one has $T\varphi(\xi) = [\varphi\circ\gamma(t)]_{\varphi(b)} \in T_{\varphi(b)}M$. Hence, $T\varphi$ is a a homomorphism

$$T\varphi : \mathcal{V}(N) \to \mathcal{V}(M)$$

and $\varphi^*\beta$ (which is related to the composition $\beta\circ T\varphi : \mathcal{V}(N) \to \mathcal{E}(M)$ is $\mathcal{V}(N) \ni Y \longmapsto \varphi^*\beta(Y) \in \mathcal{E}(N)$ with $\varphi^*\beta(Y)(b) = \beta_{\varphi(b)}(T_b\varphi(Y(b))$, $b \in N$. Hence, it is a one form $\varphi^*\beta \in \Omega^1(N)$.[*]]

According to the preceding proposition one has

$$\pi^*\alpha_k - \pi^*\alpha_j = \pi^*\left(\frac{1}{2\pi i}\,\frac{dg_{jk}}{g_{jk}}\right).$$

Moreover, one can show

(4.4) <u>LEMMA:</u> $\varphi_k^*\left(\frac{dz}{z}\right) - \varphi_j^*\left(\frac{dz}{z}\right) = \pi^*\left(\frac{dg_{jk}}{g_{jk}}\right).$[**]

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* Übung: Describe $\varphi^*\beta$ in local coordinates, also for a k-form β.

** Check!

Here, $\frac{dz}{z}$ is an abbreviation of $pr_2^* \frac{dz}{z}$ on U_j, U_k, or simply $\frac{1}{z} dz$ on $U_j \times \mathbb{C}^\times$.

As a consequence of this technical result, the expressions

$$\pi^* \alpha_j + \frac{1}{2\pi i} \varphi_j^* \left(\frac{dz}{z} \right), \quad \pi^* \alpha_k + \frac{1}{2\pi i} \varphi_k^* \left(\frac{dz}{z} \right)$$

agree on $L^\times_{U_{jk}}$ and define a global 1 form $\alpha \in \Omega^1(L^\times)$ by

$$\alpha|_{L^\times_{U_j}} := \pi^* \alpha_j + \frac{1}{2\pi i} \varphi_j^* \left(\frac{dz}{z} \right), \quad j \in I,$$

Let $m_c : L^\times \to L^\times$ be the multiplication by $c \in \mathbb{C}^*$ $m_c(\ell) = \ell c = c\ell \; (= \Psi(\ell, c))$, $\ell \in L^\times$, and for $q \in \mathbb{C} = \text{Lie } \mathbb{C}^\times$ let $\eta_q = \tilde{q} : L^\times \to TL^\times$ be the fundamental vector field

$$\eta_q(\ell) := \frac{d}{dt} \left(e^{2\pi i q t} \ell \right) \Big|_{t=0} = \left[e^{2\pi i q t} \ell \right]_\ell \in T_\ell L^\times .$$

(4.5) <u>PROPOSITION</u>: Let $\alpha \in \Omega^1(L^\times)$ be defined as above. Then

$$\nabla_X s = 2\pi i \; s^* \alpha(X) s.$$

In addition, $\alpha \in \Omega^1(M)$ satisfies,

$$\begin{aligned} &[I1] \quad \alpha(\eta_q) = q \quad \text{for all } q \in \mathbb{C}, \\ &[I2] \quad m_c^* \alpha = \alpha \quad \text{for all } c \in \mathbb{C}^\times. \end{aligned}$$

Conversely, every $\alpha \in \Omega^1(L^\times)$ with [I1], [I2] defines a connection on L by

$$\nabla_X s := 2\pi i \; s^* \alpha(X) s, \quad s \in \Gamma(U, L), \; X \in \mathcal{V}(U).$$

□ Proof: We confirm the formula $\nabla_X s = 2\pi i\ s^*\alpha(X)s$ and leave the rest as an exercise: We can restrict the consideration to the case $s \in \Gamma(U_j, L)$. We have $s = f s_j$, where $s_j(a) := \varphi_j^{-1}(a,1)$ and $f \in \mathcal{E}(U_j)$. We know

$$\nabla_X s = \left(L_X f + 2\pi i\, \alpha_j(X) f\right) s_j .$$

Now, $s^*\alpha = s^*\pi^*\alpha_j + s^* \frac{1}{2\pi i}\varphi_j^*\left(\frac{dz}{z}\right) = (\pi\circ s)^*\alpha_j + (\varphi_j\circ s)^*\left(\frac{1}{2\pi i}\frac{dz}{z}\right)$.
We know $\pi\circ s = id_{U_j}$ and $\varphi_j\circ s(a) = (a, f(a))$, $a\in U_j$.
We assume U_j to be a coordinate neighbourhood so that a vector field $X \in \mathcal{V}(U_j)$ has the form $X: U_j \to U_j\times\mathbb{R}^n$ $a \mapsto (a, V(a))$. We conclude

$$(\varphi_j\circ s^*)\frac{dz}{z}(X)(a) = \frac{dz}{f(a)}\left(D(\varphi_j\circ s).V(a)\right)$$

with $D(\varphi_j\circ s) = \begin{pmatrix} E_n \\ \operatorname{grad} f\end{pmatrix}$ the Jacobi matrix, and hence

$$D(\varphi_j\circ s)\,V(a) = \langle \operatorname{grad} f, V(a)\rangle = L_X f(a).$$

Therefore

$$2\pi i\ s^*\alpha(X) = 2\pi i\left(\alpha_j(X) + \frac{1}{2\pi i}\frac{L_X f}{f}\right), \text{ i.e}$$

$$2\pi i\ s^*\alpha(X)s = \left(2\pi i\ \alpha_j(X) f + L_X f\right) s_1 = \nabla_X s. \quad \square$$

We now have at least 3 different ways to define the concept of a connection (∇ in 4.1, (α_j) in 4.3 and α in 4.5). In order to recognise the geometric nature of the connection we introduce connections in principal fibre bundles. In this way we can regard connections on line bundles in the framework of general connections: Some parts become more complicated but others look simpler when viewed in the general case.

Principal connections.

We now let $\pi : P \to M$ be a principal fibre bundle with structure group G. G is a Lie group and we restrict, for simplicity to matix groups, i.e. to closed subgroups of $GL(k, \mathbb{C})$. We have a right action of G on P, that is a differentiable map

$$\Psi : P \times G \longrightarrow P \quad , \quad (p,g) \longmapsto \Psi(p,g),$$

satisfying $\Psi(p,e) = p$ ($e \in G$ the unit) and $\Psi(\Psi(p,g),h) = \Psi(p,gh)$. The action is mostly written in the form $pg = \Psi(p,g) = \Psi_g(p)$, such that the above requirements look like $pe = p$, $(pg)h = p(gh)$, resp. $\Psi_e = id_P$, $\Psi_g \circ \Psi_h = \Psi_{gh}$.

The action is compatible with π, i.e. $\pi(pg) = \pi(p)$ for all $g \in G$ and $p \in P$, and it is free, meaning that the map $G \ni g \longmapsto pg \in P$ is a diffeomorphism onto the fibre $P_a = \pi^{-1}(a)$, $a = \pi(p)$. Finally, $P \overset{\pi}{\to} G$, is locally trivial: Each $a \in M$ has an open neighborhood $U \subset M$ with a diffeomorphism

$$\varphi : P_U = \pi^{-1}(U) \longrightarrow U \times G$$

which respects π and the action Ψ: $pr_1 \circ \varphi = \pi|_{P_U}$ and $\varphi(pg) = \varphi(p)g$ for all $p \in P_U$ and $g \in G$ (where the action on $U \times G$ is the standard right action: $(a,h)g := (a,hg)$).

[18.11.09] ↑

Let $\mathfrak{g} = \mathrm{Lie}\, G$ denote the Lie algebra of G and $\exp : \mathfrak{g} \to G$ the exponential map.
In the case of $G = \mathbb{C}^\times$ we have $\mathbb{C} = \mathrm{Lie}\, \mathbb{C}^\times$ with $\exp(c) := e^{2\pi i c}$.

(4.6) DEFINITION: The FUNDAMENTAL FIELD associated to $X \in \mathfrak{g}$ is the vector field $\tilde{X} \in \mathcal{V}(P)$ given by

$$\tilde{X}(p) := \frac{d}{dt}\, g \exp(tX)\Big|_{t=0} = [g \exp(tX)]_p \in T_p P$$

See our case of $\mathbb{C}^\times$ and $\mathbb{C}$: $X = q \in \mathbb{C}$ & $\tilde{X} = \eta_q$.

For each point $p \in P$ the vector $\tilde{X}(p) \in T_p P$ points in the direction of the fibre P_a, $a = \pi(p)$, which can be expressed by $\tilde{X}(p) \in \mathrm{Ker}\, T_p\pi \subset T_p P$.

$$T_p\pi(\tilde{X}(p)) = [\pi(p\exp(Xt))]_a = 0,$$

since $\pi(p\exp(Xt)) = \pi(p) = 0$ for $t \in \mathbb{R}$. We also have for each $a \in M$ and $p \in P_a$: $T_p(P_a) = \mathrm{Ker}\, T_p\pi$. We call $V := \mathrm{Ker}\, T\pi \subset TP$ the VERTICAL BUNDLE. The inclusion $V \subset TP$ induces, indeed, the structure of a real vector bundle on M of dimension $\dim G$, where the projection $V \to M$ is the restriction of the canonical projection $\tau : TP \to M$. The fibres are

$$V_p = \mathrm{Ker}\, T_p\pi = T_p P_a \;, \quad a = \pi p.$$

Since the above mentioned map $X \mapsto \tilde{X}(p) \in \operatorname{Ker} T_p\bar{\pi}$ is $\mathbb{R}$-linear and injective ($\tilde{X}(p) = 0$ means $p\exp(Xt)$ is constant, hence $X = 0$) its image is all of V_p and we have another description of V_p:

$$V_p = \{ \tilde{X}(p) : X \in \mathfrak{g} \}$$

(We have used that the dimension of $\mathfrak{g}$ as an $\mathbb{R}$-vector space is $\dim G$.). As a result, $X \mapsto \tilde{X}(p)$ has an inverse $\sigma_p : V_p \to \mathfrak{g}$, and the map

$$\sigma : V \longrightarrow P \times \mathfrak{g} \quad , \sigma(v) := (\tau(v), \sigma_p(v)) , \quad v \in V,$$

turns out to be a diffeomorphism with $pr_1 \circ \sigma = \tau$ and $\sigma_p : V_p \to \{p\} \times \mathfrak{g}$ linear. Therefore, σ is a vector bundle isomorphism, and V is a trivial vector bundle.

(4.7) <u>PROPOSITION:</u> The vertical bundle $V \subset TP$ is a vector subbundle of TP and V is a trivial vector bundle.

Now the concept of a connection form on P:

(4.8) <u>DEFINITION:</u> A CONNECTION FORM on P is a one form $\alpha \in \Omega^1(P, \mathfrak{g}) = \Omega^1(P) \otimes \mathfrak{g} = \operatorname{Hom}_{\mathcal{E}(P)}(\mathcal{V}(P), \mathcal{E}(P, \mathfrak{g}))$

[I1] $\quad \alpha(\tilde{X}) = X \quad$ for all $X \in \mathfrak{g}$.

[I2] $\quad \Psi_g^* \alpha = g^{-1} \alpha g \quad$ for all $g \in G$.

Here, $\Psi_g : P \to P$ is the diffeomorphism induced by Ψ : $\Psi_g(p) = pg$, $\Psi_g : p \longmapsto pg$. And $g^{-1}\alpha g$ is well-defined since for a matrix group G every $g \in G$ induces a map $\mathfrak{g} \to \mathfrak{g}$, $X \longmapsto g^{-1}Xg$ (which is $\mathrm{Ad}_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$).

Evidently, the conditions [I1], [I2] in (4.5) and in (4.8) agree, since $c^{-1}\alpha c = \alpha$ in the case of $c \in \mathbb{C}^\times = G$.

Given a principal fibre bundle $P \xrightarrow{\pi} M$ with a connection form $\alpha \in \Omega^1(M)$ we obtain an associated HORIZONTAL BUNDLE $H \subset TP$ in the following way:

$$H := \operatorname{Ker} \alpha \subset TP ,$$

with the fibres $H_p = \operatorname{Ker} \alpha_p = \{ \xi_p \in T_pP \mid \alpha_p(\xi_p) = 0 \}$. By the induced structure from TP the set H is indeed a real vector bundle of dimension $\dim M$. We have $H \cap V = \{0\}$, i.e. $H_p \cap V_p = \{0\}$ for all $p \in P$: In order to show this, let $\xi \in H_p \cap V_p$. Then $\alpha(\xi) = 0$ and $\xi = \tilde{X}(p)$ for a suitable $X \in \mathfrak{g}$. Hence $\alpha(\tilde{X}(p)) = X = 0$ by [I1] and therefore $\xi = 0$.

As a consequence, we obtain the decomposition

$$TP = H \oplus V$$

of TP into the direct sum of two real vector subbundles H, V of TP. The action Ψ_g induces (by [I2]) an isomorphism $T_p\Psi_g : H_p \to H_{pg}$ for all $(p,g) \in P \times G$.

We have shown the first half of the following

(4.9) PROPOSITION: A connection form $\alpha \in \Omega^1(P)$ on a principal fibre bundle $P \to M$ defines the horizontal bundle $H := \operatorname{Ker} \alpha$ with

$$[H1] \quad TP = H \oplus V$$
$$[H2] \quad T_p \tau_g (H_p) = H_{pg} \quad \text{for all } (p,g) \in P \times G .$$

Conversely, any vector subbundle $H \subset TP$ with [H1] and [H2] induces a connection form $\alpha \in \Omega^1(M)$ with $H = \operatorname{Ker} \alpha$.

□ Proof. The decomposition defines a projection $v : TP \to TP$ which fibrewise is the linear projection $v_p : T_pP \longrightarrow T_pP$ with $\operatorname{Ker} v_p = H_p$ and $\operatorname{Im} v_p = V_p$. v is a vector bundle homomorphism and $\alpha := \sigma \circ v$ as the map

$$p \longmapsto \alpha_p = \sigma_p \circ v_p \in \operatorname{End}_{\mathbb{R}}(T_pP, \mathfrak{g})$$

defines the connection form $\alpha \in \Omega^1(P, \mathfrak{g})$ with $H = \operatorname{Ker} \alpha$. □

Exploiting the preceding proof we obtain another description of a principal connection (given by a one form α with [I1], [I2], or a decomposition $H \oplus P$ of TP with [H2]):

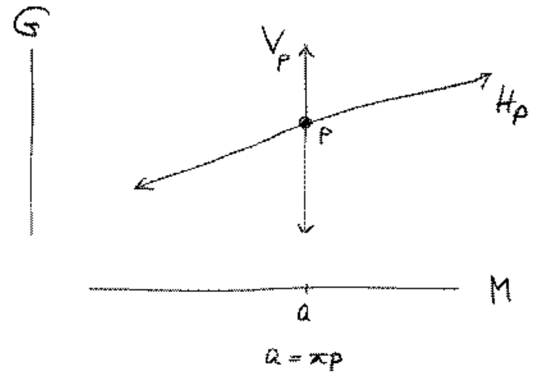
(4.10) <u>PROPOSITION</u>: A principal connection is also given by a vector bundle projection onto V

$$v : TP \to TP \,,$$

which is Ψ-invariant, i.e. $T\Psi_g \circ v = v \circ T\Psi_g$ for each $g \in G$.

☐ <u>Proof</u>. Being a projection onto V means: v is a vector bundle homomorphism with $v \circ v = v$ and $\operatorname{Im} v = V$. Hence $H := \operatorname{Ker} v \subset TP$ is a complementary vector bundle ($TP = H \oplus V$) with $T\Psi_g : H_p \to H_{pg}$ an isomorphism ([H1] and [H2]).
Conversely, a decomposition $TP = H \oplus V$ with [H1] and [H2] immediately yields the projection $v : TP \to TP$ onto V satisfying $T\Psi_g \circ v = v \circ T\Psi_g$. ☐

The concept of a horizontal bundle $H \subset TP$ complementing the vertical bundle is GEOMETRIC in nature: At each point $p \in P$ a horizontal space $H_p \subset T_pP$ is assigned which is transverse to the vertical space V_p. Sketch:



In physics, P is called the space of phase factors, and α is the (global) gauge potential.

We obtain local gauge potentials by pullback: Let $\sigma : U \to P$ be a section, i.e. smooth and $\sigma \circ \pi = id_U$ ($U \subset M$ open). Denote

$$A^\sigma := \sigma^* \alpha \in \Omega^1(U, \mathfrak{g}) .$$

Then A^σ is called a local gauge potential (given by σ). How do these local gauge potentials fit together?

(4.11) <u>PROPOSITION</u>: Given two sections $\sigma, \sigma' : U \to P$ over $U \subset P$ the corresponding local gauge potentials

$$A = \sigma^* \alpha \ , \ A' = \sigma'^* \alpha$$

satisfy

$$A' = g A g^{-1} + g dg^{-1} ,$$

where $g(a) \in G$ is the uniquely defined group element with $\sigma(a) = \sigma'(a) g(a)$.

□ <u>Proof</u>. Let $Y \in T_a M$ be given by the curve $\gamma(t)$, i.e. $Y = [\gamma]_a$. Then

$$\sigma'^* \alpha(Y)(a) = \alpha_{\sigma'(a)}\left(\frac{d}{dt} \sigma' \circ \gamma(t)\Big|_{t=0}\right),$$

and

$$\frac{d}{dt} \sigma' \circ \gamma(t)\Big|_{t=0} = \frac{d}{dt} (\sigma \cdot g^{-1}) \circ \gamma(t)\Big|_{t=0} = \frac{d}{dt} \left(\sigma \circ \gamma(t)\right)\left(g^{-1} \circ \gamma(t)\right)\Big|_{t=0}$$

$$= \frac{d}{dt} \sigma \circ \gamma(t) g^{-1}(a)\Big|_{t=0} + \frac{d}{dt} \sigma(a) \cdot g^{-1} \circ \gamma(t)\Big|_{t=0} .$$

With $h := \bar{g}^1(a) \in G$, the first term is

$$\frac{d}{dt}\, \Psi_h \circ \sigma \circ \gamma(t)\,\Big|_{t=0} = T_p\Psi_h\big(T_a\sigma(Y)\big) \quad , \; p = \sigma(a).$$

To analyze the second term $\frac{d}{dt}\, \sigma(a)\cdot \bar{g}^1\circ\gamma(t)\big|_{t=0} = \frac{d}{dt}\, \sigma(a)\, g(a)\, \bar{g}^1\circ\gamma(t)\big|_{t=0}$ we observe that $X := \frac{d}{dt}\, g(a)\, \bar{g}^1(\gamma(t))\,\big|_{t=0}$ is a tangential vector $X \in T_eG$, since $g(a)\bar{g}^1(\gamma(0)) = e =$ identity, and can be viewed as to be a Lie algebra element $X \in \mathfrak{g} \cong T_eG$. Since $g(a)\bar{g}^1(\gamma(t))$ is the matrix multiplication: $X = g(a)\frac{d}{dt}\,\bar{g}^1(\gamma(t))\,\big|_{t=0} = g(a)\, T_a\bar{g}^1(Y)$. For every curve $\eta(t)$ in G with $\eta(0) = e$ and $[\eta]_e = X$ one has

$$\tilde{X}(p') = \frac{d}{dt}\big(p'\eta(t)\big)\big|_{t=0} \quad , \quad p' \in P,$$

in particular for $\eta(t) = g(a)\,\bar{g}^1(\gamma(t))$:

$$\tilde{X}(p') = \frac{d}{dt}\, p'\, g(a)\, \bar{g}^1(\gamma(t))\big|_{t=0}$$

According to [I1] we have $\alpha_{p'}\big(\tilde{X}(p')\big) = X$ and we conclude $(p' = \sigma'(a))$:

$$(1) \quad \alpha_{p'}\Big(\frac{d}{dt}\,\sigma'(a)\, g(a)\, \bar{g}^1(\gamma(t))\Big)\Big|_{t=0} = X = g(a)\, T_a\bar{g}^1(Y).$$

Moreover, for the first term $T_p\Psi_h(T_a\sigma(Y))$ the condition [I2] reads

$$\alpha_{ph}\Big(T_p\Psi_h\big(T_a\sigma(Y)\big)\Big) = h^{-1}\,\alpha_p\big(T_a\sigma(Y)\big)\, h,$$

and because of $p' = ph$, $h^{-1} = g(a)$ and $A_a(Y) = \sigma^*\alpha(Y) = \alpha_p\big(T_a\sigma(Y)\big)$ we obtain

$$(2) \quad \alpha_{p'}\big(T_p\Psi_h(T_a\sigma(Y)\big) = g(a)\, A_a(Y)\, \bar{g}^1(a)$$

Putting everything together we have

$$A'_a(Y) = \alpha_{p'}\left(\tfrac{d}{dt}\, \sigma'\circ\gamma(t)\right)\Big|_{t=0} = \omega_{p'}\left(T_p\Psi_u(T_a\sigma(Y)) + \tfrac{d}{dt}\, \sigma(a)\bar{g}^{1}\circ\gamma(t)\Big|_{t=0}\right)$$

$$\underset{(2)\,\&\,(1)}{=} g(a)\, A_a(Y)\, \bar{g}^{1}(a) + g(a)\, T_a \bar{g}^{1}(Y)$$

$\square$

As before, local trivializations (here given by local sections) over U_j for an open cover (U_j) of M yield local gauge potentials A_j with a transition rule and vice versa. In detail:

Let $\sigma_j : U_j \to P$ smooth sections with its corresponding trivialization $\varphi_j : P_{U_j} \to U_j \times G$, $p \mapsto (\pi p, \hat{\sigma}_j(p))$, where $\hat{\sigma}_j(p) \in G$ is given by $p = \sigma_j(\pi(p))\, \hat{\sigma}_j(p))$. These data induce transition functions $g_{jk} \in \mathcal{E}(U_{jk}, G)$, given by $g_{jk} = \hat{\sigma}_k \circ \hat{\sigma}_j^{1}$. In particular, (g_{jk}) satisfies [C], and

$$\sigma_j = \sigma_k\, g_{jk} \quad \text{on } U_{jk} \neq \emptyset .$$

Let $A_j := \sigma_j^* \alpha \in \Omega^1(U_j, \mathfrak{g})$.

(4.12) <u>PROPOSITION</u>: Let α be a connection form on the principal fibre bundle $P \to M$. Define the local gauge potentials A_j and the transition functions g_{jk} as above (depending on the sections σ_j). Then

$$[Z] \quad A_k = g_{jk}\, A_j\, \bar{g}_{jk}^{1} + g_{jk}\, d\bar{g}_{jk}^{1} .$$

Conversely, a collection (A_j) of 1forms with [Z] defines a connection form α whose local gauge potentials are the A_j (Exercise[*]).

Note that [Z] is the same as [Z] above for the case $G = \mathbb{C}^\times$ since $dg\,g^{-1} + g\,dg^{-1} = 0$ in general and $g\alpha g^{-1} = \alpha$ for $g \in \mathbb{C}^\times$.

We collect our results for the special case of a line bundle L with its corresponding principal fibre bundle $L^\times \subset L$:

(4.13) <u>PROPOSITION:</u> A connection on a line bundle $L \to M$ is given by one of the following five equivalent data:

1° A covariant derivative $\nabla_X : \Gamma(U, L) \to \nabla_X(U, L)$ with [K1] and [K2] according to (4.1)

2° A collection of 1-forms $(\alpha_j)_{j \in I}$, $\alpha_j \in \Omega^1(U_j)$, satisfying [Z], where $(U_j)_{j\in I}$ is an open cover where L_{U_j} is trivial; see (4.3) and (4.8).

3° A connection form $\alpha \in \Omega^1(L^\times)$ with [I1] and [I2]; see (4.5) and (4.12).

4° A vector subbundle $H \subset TL^\times$ such that [H1] and [H2]; cf. (4.9).

5° A vector bundle homomorphism $\sigma : TL^\times \to TL^\times$ which is a T_g-invariant projection onto the vertical bundle $V \subset TL^\times$; cf. (4.10).

We illustrate 1° – 5° in the case of the trivial line bundle $L = M \times \mathbb{C}$ with global coordinates $q^1, \dots, q^n$ for M (e.g. $M \subset \mathbb{R}^n$ open):

1° ∇ is given by $\alpha = \alpha_j \, dq^j \in \Omega^1(M)$, $\alpha_j \in \mathcal{E}(M, \mathbb{C})$, in such a way that ∇_X for $X = X^j \frac{\partial}{\partial q^j}$ is

$$\begin{aligned}\nabla_X f s_1 &= \left(L_X f + 2\pi i \, \alpha(X) f \right) s_1 \\ &= \left(X^j \frac{\partial f}{\partial q^j} + 2\pi i \, \alpha_j X^j f \right) s_1\end{aligned}$$

or

$$\nabla_X \left(a, f(a)\right) = \left(a, X^j \left(\frac{\partial f}{\partial q^j} + 2\pi i \alpha_j f \right)\right)$$

for $f \in \mathcal{E}(M)$, where $s_1(a) = (a, 1)$, $a \in M$.

2° is the same, since $U_j = M$.

3° The connection form on $L^\times$ is

$$\alpha + \frac{1}{2\pi i} \frac{dz}{z} = \alpha_j \, dq^j + \frac{1}{2\pi i} \frac{dz}{z}$$

4° The horizontal space $H_p \in T_p L^\times \cong \mathbb{R}^n \times \mathbb{C}$ in $p = (a, w) \in L^\times$ is

$$H_p = \left\{ (X, Z) \in \mathbb{R}^n \times \mathbb{C} \;\middle|\; \alpha_j(a) X^j + \frac{1}{2\pi i w} Z = 0 \right\}.$$

Since the coefficient $\frac{1}{2\pi i w}$ does not vanish ($w \in \mathbb{C}^\times$!), the kernel $H_p = \ker \left(\alpha + \frac{1}{2\pi i} \frac{dz}{z} \right)_p$ has real dimension n and we have $T_p \mathbb{C}^\times \cong \mathbb{R}^n \times \mathbb{C} = H_p \oplus \left(\{0\} \times \mathbb{C} \right)$.

5° is essentially the same as 4°.

In our Proposition (4.13) we have already anticipated that there is also a general procedure in which a given connection on the principal fibre bundle $L^\times$ induces a connection on the line bundle L: One simply uses the globally given connection form $\alpha \in \Omega^1(L^\times)$ and local sections $s_j : U_j \to L$ (where $(U_j)_{j\in I}$ is an open cover) to obtain $\alpha_j := s_j^* \alpha \in \Omega^1(U_j)$ satisfying [Z] and thus defining a connection on L.

This procedure has its generalization to associated vector bundles of a principal fibre bundle $P \to M$ with connection.

Let $\rho : G \to GL(r, \mathbb{C})$ be a representation of the structure group G of $P \to M$. The associated vector bundle E_ρ is

$$E_\rho = P \times_\rho \mathbb{C}^r \longrightarrow M ,$$

the quotient manifold of $P \times \mathbb{C}^r$ with respect to the equivalence relation

$$(p, v) \sim (p', v') \iff \exists g \in G : (p', v') = (pg, \rho(g^{-1})v):$$

$$E_\rho := P \times \mathbb{C}^r / \sim .$$

Assume that the connection on $P \to M$ is given by a connection one form $\alpha \in \Omega^1(P)$. Let $(U_j)_{j\in I}$ be an open cover of M with sections $\sigma_j : U_j \to P$, and define α_j by $\alpha_j := \rho_*(\sigma_j^* \alpha) \in \Omega^1(U_j, \mathbb{C}^r)$. Here, the representation $\mathrm{Lie}\,\varrho : \mathfrak{g} \longrightarrow \mathfrak{g}(\mathbb{C}^r) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^r)$ is induced by ϱ:

$$\mathrm{Lie}\,\varrho(X) := \frac{d}{dt}\, \varrho(\exp tX)\Big|_{t=0} \in \mathfrak{g}(\mathbb{C}^r).$$

And it leads to the definition $\varrho_* \beta(Y) := \mathrm{Lie}\,\varrho(\beta(Y)) \in \mathbb{C}^r$ for a $\mathfrak{g}$-valued form $\beta \in \Omega^1(U, \mathfrak{g})$ and $Y \in \mathcal{V}(U)$. The collection (α_j) satisfies the compatibility condition

$$[Z_\varrho] \qquad \alpha_k = \tilde{g}_{jk}\, \alpha_j\, \tilde{g}_{jk}^{-1} + \tilde{g}_{jk}\, d\tilde{g}_{jk}^{-1},$$

where $\tilde{g}_{jk} := \varrho(g_{jk})$. These (α_j) induce (as in (4.8)) a covariant derivative

$$\nabla_X : \Gamma(U, E_\varrho) \to \Gamma(U, E_\varrho) \quad , U \subset M \text{ open},$$

compatible with restrictions to $V \subset U$ and satisfying [K1] and [K2] with L replaced by E_ϱ.

The description of a connection on a line bundle and on more general bundles is now complete. The geometric nature of this concept has been neglected so far except for the decomposition of the tangent bundle of the total bundle into its vertical and a horizontal sub-bundle. The geometry of connections on a line bundle is the subject of the next three sections where we investigate parallel transport, curvature and hermitean structure.

We conclude this section by discussing the existence of connections and the question of the structure of all connections.

(4.14) PROPOSITION: On every line bundle over a (paracompact) manifold there exists a connection.

□ Proof. We take an open cover $(\mathcal{U}_j)_{j\in I}$ over which a given line bundle $L \to M$ is trivial with trivializations $\varphi_j : L_{\mathcal{U}_j} \to \mathcal{U}_j \times \mathbb{C}$ and local sections $s_j(a) = \varphi_j^{-1}(a,1)$, $a \in \mathcal{U}_j$. We choose $\beta_j \in \Omega^1(\mathcal{U}_j)$ and obtain the connections $\nabla_X^{(j)} f s_j := (L_X f + 2\pi i \beta_j(X) f) s_j$ on $L_{\mathcal{U}_j}$, $j \in I$. Let (κ_j) be a partition of unity subordinate to $(\mathcal{U}_j)$, i.e $\kappa_j \in \mathcal{E}(M)$, $\operatorname{Supp} \kappa_j \subset \mathcal{U}_j$, (κ_j) locally finite and $\sum_{j\in I} \kappa_j(a) = 1$ for each $a \in M$. Then

$$\nabla_X s := \sum_{i\in I} \kappa_i \nabla_X^{(i)} s|_{\mathcal{U}_i \cap \mathcal{U}} \quad , s \in \Gamma(\mathcal{U}, L),$$

defines a connection on L. □

The sum of 2 connections ∇, ∇' on $L \to M$ is in general not a connection. The difference $\nabla - \nabla'$ is a one form on M. In fact, for $X \in \mathcal{V}(M)$ and $s \in \Gamma(M, L)$ the equation

$$(\nabla_X - \nabla'_X) s = \beta(X) s$$

defines a value $\beta(X)(a) \in \mathbb{C}$ for $s(a) \neq 0$. Because of

$$(\nabla_X - \nabla'_X) f s = L_X f s + f \nabla_X s - L_X f s - f \nabla'_X s = f (\nabla_X - \nabla'_X) s$$

this value is independent of s. Since for every $a \in M$ there exists $s \in \Gamma(M, L)$ with $s(a) \neq 0$ we obtain a uniquely defined $\beta(X) \in \mathcal{E}(M)$ such that $X \longmapsto \beta(X)$ is $\mathcal{E}(M)$-linear, and hence $\beta \in \Omega^1(M)$.

(4.15) <u>PROPOSITION:</u> Given a fixed connection ∇ on L, every other connection on L has the form

$$\nabla' = \nabla + \beta$$

for an arbitrary $\beta \in \Omega^1(M)$. The set of connections is the affine space $\nabla_X + \Omega^1(M)$. It can be understood as an affine subspace of $\Omega^1(L^*)$.

□ <u>Proof</u>. It only remains to check that $\nabla_X + \beta$, $\beta \in \Omega^1(M)$, is a connection. The last statement follows from (4.8). □

Isomorphism classes of line bundles with connections are classified by cohomology, similar to the description of $\mathrm{Pic}^\infty(M) \cong H^1(M, \mathcal{E}^\times)$, now using (g_{ij}) and (α_j). Details are in [BRYLINSKI].

General references for connections on principal bundles and their associated bundles are [KOBAYASHI – NOMIZU] and [BAUM].